Propagation of Electromagnetic Waves in Linear Media and Pseudo-Hermiticity

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We express the electromagnetic field propagating in an arbitrary time-independent non-dispersive medium in terms of an operator that turns out to be pseudo-Hermitian for Hermitian dielectric and magnetic permeability tensors $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$. We exploit this property to determine the propagating field. In particular, we obtain an explicit expression for a planar field in an isotropic medium with $\overleftrightarrow{\varepsilon}=\varepsilon\overleftrightarrow{1}$ and $\overleftrightarrow{\mu}=\mu\overleftrightarrow{1}$ varying along the direction of the propagation. We also study the scattering of plane waves due to a localized inhomogeneity.

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The study of electromagnetic (EM) fields propagating in a linear (non-dispersive) medium is one of the oldest problems of physics. Most textbook treatments of this problem begin with the assumption of harmonic time-dependence. In this letter, we present a systematic solution of this problem that does not rely on this assumption and instead makes use of the notion of "pseudo-Hermitian operator" that was originally developed in [1, 2] to deal with $\mathcal{PT}$-symmetric Hamiltonians [3].[14]

Consider the propagation of EM fields in a linear source-free medium with time-independent dielectric and inverse magnetic permeability tensors $\overleftrightarrow{\varepsilon}=\overleftrightarrow{\varepsilon}(\vec{x})$ and $\overleftrightarrow{\mu}'=\overleftrightarrow{\mu}'(\vec{x})$, [5]. Then the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, satisfy Maxwell's equations:

$$\vec{\nabla}\cdot\vec{D}=0,\qquad \vec{\nabla}\cdot\vec{B}=0, \tag{1}$$

$$\dot{\vec{B}}+\mathfrak{D}\vec{E}=0,\qquad \dot{\vec{D}}-\mathfrak{D}\vec{H}=0, \tag{2}$$

where $\vec{D}:=\overleftrightarrow{\varepsilon}\,\vec{E}$, $\vec{H}:=\overleftrightarrow{\mu}'\vec{B}$, a dot means a time-derivative, and $\mathfrak{D}$ denotes the curl operator; $\mathfrak{D}\vec{F}:=\vec{\nabla}\times\vec{F}$ for any vector field $\vec{F}$. Our aim is to solve (2) for $\vec{E}=\vec{E}(x,t)$ and $\vec{B}=\vec{B}(\vec{x},t)$ in terms of the initial fields $\vec{E}_0:=\vec{E}(\vec{x},0)$ and $\vec{B}_0:=\vec{B}(\vec{x},0)$. We will consider lossless media for which $\overleftrightarrow{\varepsilon}(\vec{x})$ and $\overleftrightarrow{\mu}'(\vec{x})$ are (Hermitian) positive-definite matrices for all $\vec{x}\in\mathbb{R}^3$.

We begin our study by expressing Eqs. (2) in terms of $\vec{E}$ and $\vec{B}$. Taking the time-derivative of the second of these equations and using the result in the first, we obtain $\ddot{\vec{E}}+\Omega^2\vec{E}=0$ where

$$\Omega^2:=\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'\mathfrak{D}. \tag{3}$$

We can easily solve $\ddot{\vec{E}}+\Omega^2\vec{E}=0$ to find

$$\vec{E}(\vec{x},t)=\cos(\Omega t)\vec{E}_0(\vec{x})+\Omega^{-1}\sin(\Omega t)\dot{\vec{E}}_0(\vec{x}), \tag{4}$$

where $\dot{\vec{E}}_0(\vec{x}):=\dot{\vec{E}}(\vec{x},0)=\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'\vec{B}_0(\vec{x})$ and

$$\cos(\Omega t):=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,(t^2\Omega^2)^n, \tag{5}$$

$$\Omega^{-1}\sin(\Omega t):=t\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(t^2\Omega^2)^n. \tag{6}$$

In view of the first equation in (2), the magnetic field is given by $\vec{B}(\vec{x},t)=\vec{B}_0(\vec{x})-\int_0^t ds\,\mathfrak{D}\vec{E}(\vec{x},s)$, [15].

Relation (4) is of limited practical importance, because in general its right-hand side involves infinite derivative expansions. We can choose the initial fields such that they are eliminated by a positive integer power of Ω^2. This leads to an infinite class of exact solutions of Maxwell's equations with polynomial time-dependence. In order to use (4) in dealing with physically more interesting situations, we will express (5) and (6) as integral operators and compute the corresponding integral kernels (propagators).[16] This requires a closer look at Ω^2.

Let $\prec\vec{F},\vec{G}\succ:=\int_{\mathbb{R}^3}dx^3\vec{F}(\vec{x})^*\cdot\vec{G}(\vec{x})$, where $\vec{F}$ and $\vec{G}$ are vector fields. Then $\mathcal{H}:=\{\vec{F}:\mathbb{R}^3\to\mathbb{C}^3|\prec\vec{F},\vec{F}\succ<\infty\}$ together with the inner product $\prec\cdot,\cdot\succ$ form a Hilbert space. It is easy to check that the curl operator $\mathfrak{D}$ is actually a Hermitian operator acting in $\mathcal{H}$. That is $\prec\vec{F},\mathfrak{D}\vec{G}\succ=\prec\mathfrak{D}\vec{F},\vec{G}\succ$. The same is true about $\mathfrak{D}\,\overleftrightarrow{\mu}'\mathfrak{D}$. But Ω^2 is not Hermitian. Its adjoint $\Omega^{2\dagger}$, which is defined by the condition $\prec\vec{F},\Omega^2\vec{G}\succ=\prec\Omega^{2\dagger}\vec{F},\vec{G}\succ$, satisfies $\Omega^{2\dagger}=\overleftrightarrow{\varepsilon}\Omega^2\overleftrightarrow{\varepsilon}^{-1}$. This means that $\Omega^2:\mathcal{H}\to\mathcal{H}$ is an $\overleftrightarrow{\varepsilon}$-pseudo-Hermitian operator [1]. Here we view $\overleftrightarrow{\varepsilon}$ as an operator acting in $\mathcal{H}$ according to $(\overleftrightarrow{\varepsilon}\vec{F})(\vec{x}):=\overleftrightarrow{\varepsilon}(\vec{x})\vec{F}(\vec{x})$. Indeed, because $\overleftrightarrow{\varepsilon}(\vec{x})$ is a positive-definite matrix for all $\vec{x}$, the operator $\overleftrightarrow{\varepsilon}$ is a positive-definite (metric) operator. This in turn implies that it defines a new positive-definite inner product that renders Ω^2 self-adjoint [1]; letting $\prec\vec{F},\vec{G}\succ_\varepsilon:=\prec\vec{F},\overleftrightarrow{\varepsilon}\vec{G}\succ$, we find $\prec\vec{F},\Omega^2\vec{G}\succ_\varepsilon=\prec\Omega^2\vec{F},\vec{G}\succ_\varepsilon$. Furthermore, Ω^2 may be mapped to a Hermitian operator $h:\mathcal{H}\to\mathcal{H}$ via a similarity transforms [2]. A possible choice for h is [6]

$$h:=\overleftrightarrow{\varepsilon}^{\frac{1}{2}}\Omega^2\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}=\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}\mathfrak{D}\,\overleftrightarrow{\mu}'\mathfrak{D}\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}. \tag{7}$$

Note that because $\overleftrightarrow{\varepsilon}(\vec{x})$ and $\overleftrightarrow{\mu}'(\vec{x})$ are assumed to be positive matrices for all $\vec{x}$, they have a unique positive square root [7]. This in turn implies that h is a positive operator with a nonnegative spectrum and a unique positive square root $h^{\frac{1}{2}}$, [17].

Because h is Hermitian, we can use its spectral resolution to compute any function Φ of h, [8]. In light of

$\Phi(\Omega^2)=\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}\Phi(h)\overleftrightarrow{\varepsilon}^{\frac{1}{2}}$, this allows for the calculation of the action of $\Phi(\Omega^2)$ on any vector field $\vec{G}$:

$$\Phi(\Omega^2)\vec{G}(\vec{x})=\int_{\mathbb{R}^3}dy^3\,\overleftrightarrow{\varepsilon}(\vec{x})^{-\frac{1}{2}}\langle\vec{x}|\Phi(h)|\vec{y}\rangle\,\overleftrightarrow{\varepsilon}(\vec{y})^{\frac{1}{2}}\vec{G}(\vec{y}),\quad(8)$$

where we have used Dirac's bra-ket notation.

To demonstrate the effectiveness of the above method we consider the textbook [9] problem of the planar propagation of the initial fields

$$\vec{E}_0(z)=\mathcal{E}_0(z)\,e^{-ik_0z}\,\hat{\imath},\qquad \vec{B}_0(z)=\mathcal{B}_0(z)\,e^{-ik_0z}\,\hat{\jmath},\quad(9)$$

along the z-axis in an isotropic medium with $\overleftrightarrow{\varepsilon}(\vec{x})=\varepsilon(z)\overleftrightarrow{1}$ and $\overleftrightarrow{\mu}'(\vec{x})=\mu(z)^{-1}\overleftrightarrow{1}$, where $\vec{x}=:(x,y,z)$, $\overleftrightarrow{1}$ is the identity matrix, $\mathcal{E}_0$ and $\mathcal{B}_0$ are given envelope functions, k_0 is the principal wave number at which the Fourier transform of $\vec{E}_0(z)$ and $\vec{B}_0(z)$ are picked, $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors along the x- and y-axes, and $\varepsilon(z)$ and $\mu(z)$ are respectively (the z-dependent) dielectric and magnetic permeability constants. We will in particular consider the cases that $\varepsilon(z)$ and $\mu(z)$ tend to constant values as $z\to\pm\infty$.

For this configuration all the fields are independent of x and y-coordinates and we have $\Omega^2=\varepsilon(z)^{-1}p\,\mu(z)^{-1}p$, where $p:=-i\frac{d}{dz}$,

$$h=\varepsilon(z)^{-\frac{1}{2}}p\;\mu(z)^{-1}p\;\varepsilon(z)^{-\frac{1}{2}},\quad(10)$$

and $\dot{\vec{E}}_0(z):=\varepsilon(z)^{-1}\dot{\vec{D}}_0(z)=\varepsilon(z)^{-1}\mathfrak{D}\vec{B}_0(z)=\varepsilon(z)^{-1}[ik_0\mathcal{B}_0(z)-\mathcal{B}_0'(z)]\,e^{-ik_0z}\,\hat{\imath}$.

In order to determine the spectral resolution of h we need to solve the time-independent Schrödinger equation for the position-dependent-mass Hamiltonian (10), i.e.,

$$-\varepsilon(z)^{-\frac{1}{2}}\frac{d}{dz}\left(\mu(z)^{-1}\frac{d}{dz}\left[\varepsilon(z)^{-\frac{1}{2}}\psi(z)\right]\right)=\omega^2\psi(z).\quad(11)$$

Because of the above-mentioned asymptotic behavior of $\varepsilon(z)$ and $\mu(z)$, the eigenfunctions of h are the solutions of (11) fulfilling the bounded boundary conditions at $\pm\infty$. Also note that $\omega^2\in\mathbb{R}^+$, because h is a positive operator.

For an arbitrary ε we cannot solve (11) exactly. Therefore, we employ the WKB approximation. To do this we express ψ in its polar representation: $\psi=R\,e^{iS}$ where R and S are real-valued functions. Inserting $\psi=R\,e^{iS}$ in (11) gives

$$S'(z)^2+Q(z)=\omega^2\varepsilon(z)\,\mu(z),\quad(12)$$
$$\frac{d}{dz}\left[\mu(z)^{-1}R_-(z)^2S'(z)\right]=0,\quad(13)$$

where

$$Q(z):=-\frac{[\mu(z)^{-1}R_-'(z)]'}{\mu(z)^{-1}R_-(z)},\qquad R_-(z):=\frac{R(z)}{\varepsilon(z)^{\frac{1}{2}}}.\quad(14)$$

WKB approximation amounts to neglecting $Q(z)$ in (12). This yields

$$S(z)=\omega\,u(z)+c_1,\qquad R_-(z)=c_2\,\varepsilon(z)^{-\frac{1}{4}}\mu(z)^{\frac{1}{4}},\quad(15)$$

where

$$u(z):=\int_0^z d\mathfrak{z}\;v(\mathfrak{z})^{-1},\qquad v(z):=[\varepsilon(z)\mu(z)]^{-\frac{1}{2}},\quad(16)$$

and c_1,c_2 are possibly ω-dependent integration constants. Using these choices for S and R_- and fixing c_1 and c_2 appropriately, we find the following δ-function normalized eigenfunctions for all $\omega\in\mathbb{R}$.

$$\psi_\omega(z):=\frac{e^{i\omega u(z)}}{\sqrt{2\pi\,v(z)}}.\quad(17)$$

Next, we use ψ_ω to express $\cos(h^{\frac{1}{2}}t)$ and $h^{-\frac{1}{2}}\sin(h^{\frac{1}{2}}t)$ in terms of their spectral resolution:

$$\cos(h^{\frac{1}{2}}t)=\int_{-\infty}^{\infty}d\omega\;\cos(\omega t)\;|\psi_\omega\rangle\langle\psi_\omega|,\quad(18)$$
$$h^{-\frac{1}{2}}\sin(h^{\frac{1}{2}}t)=\int_{-\infty}^{\infty}d\omega\;\frac{\sin(\omega t)}{\omega}\;|\psi_\omega\rangle\langle\psi_\omega|.\quad(19)$$

Using (17) – (19) and the identities $\int_{-\infty}^{\infty}d\omega\;e^{ia\omega}=2\pi\delta(a)$ and $\int_{-\infty}^{\infty}d\omega\;e^{ia\omega}/\omega=\pi i\,\mathrm{sign}(a)$, with δ denoting the Dirac delta function and $\mathrm{sign}(x):=x/|x|$ for $x\neq0$ and $\mathrm{sign}(0):=0$, we find

$$\begin{aligned}\langle z|\cos(h^{\frac{1}{2}}t)|w\rangle &= \int_{-\infty}^{\infty}d\omega\;\cos(\omega t)\;\psi_\omega(z)\,\psi_\omega(w)^*\\ &= \frac{1}{2}\,[v(z)v(w)]^{-\frac{1}{2}}\Delta(z,w;t),\quad(20)\\ \langle z|h^{-\frac{1}{2}}\sin(h^{\frac{1}{2}}t)|w\rangle &= \int_{-\infty}^{\infty}d\omega\,\omega^{-1}\sin(\omega t)\,\psi_\omega(z)\,\psi_\omega(w)^*\\ &= \frac{1}{4}\,[v(z)v(w)]^{-\frac{1}{2}}\Sigma(z,w;t),\quad(21)\end{aligned}$$

where

$$\Delta(z,w;t):=\delta[u(w)-u(z)+t]+\delta[u(w)-u(z)-t],$$

$$\Sigma(z,w;t):=\mathrm{sign}[u(w)-u(z)+t]-\mathrm{sign}[u(w)-u(z)-t].$$

Because u is a monotonically increasing function that vanishes only at $z=0$, it is invertible and its inverse u^{-1} is also a monotonically increasing function with a single zero at $z=0$. This implies that the quantity

$$w_\pm(z,t):=u^{-1}(u(z)\pm t)\quad(22)$$

is the only zero of $u(w)-(u(z)\pm t)$. Hence,

$$\delta[u(w)-u(z)\pm t]=\frac{\delta[w-w_\mp(z,t)]}{|u'(w_\mp(z,t))|}.$$

In view of this relation and (16), we have

$$\Delta(z,w;t)=\frac{\delta[w-w_-(z,t)]}{v(w_-(z,t))^{-1}}+\frac{\delta[w-w_+(z,t)]}{v(w_+(z,t))^{-1}}.\quad(23)$$

Furthermore, because both u and u^{-1} are monotonically increasing, for $t>0$ we have $w_-(z,t)<w_+(z,t)$ and

$$\Sigma(z,w;t)=\left\{\begin{array}{ll} 2 & \text{for} \quad w_-(z,t)<w<w_+(z,t), \\ 0 & \qquad\qquad \text{otherwise.}\end{array}\right. \qquad (24)$$

Next, we compute the action of $\cos(\Omega t)$ on an arbitrary test function $f(z)$. To do this we set $\Phi(\Omega^2)=\cos(\Omega t)$ in (8) and use (20) and (23) to evaluate the corresponding integral. This yields

$$\cos(\Omega t)f(z)=f_-(z,t)+f_+(z,t), \qquad (25)$$

where

$$f_\pm(z,t):=\frac{1}{2}\left[\frac{\varepsilon(w_\pm(z,t))v(w_\pm(z,t))}{\varepsilon(z)v(z)}\right]^{\frac{1}{2}} f(w_\pm(z,t)).$$

Similarly, setting $\Phi(\Omega^2)=\Omega^{-1}\sin(\Omega t)$ in (8) and using (21) and (24) we find

$$\Omega^{-1}\sin(\Omega t)f(z)=\frac{[\varepsilon(z)v(z)]^{-\frac{1}{2}}}{2}\int_{w_-(z,\tau)}^{w_+(z,\tau)}dw\left[\frac{\varepsilon(w)^{\frac{1}{2}}f(w)}{v(w)^{\frac{1}{2}}}\right]. \qquad (26)$$

Finally, we use (16), (25) and (26) to express (4) as

$$\begin{aligned}\vec{E}(z,t) \;=\; & \frac{1}{2}\left[\frac{\mu(z)}{\varepsilon(z)}\right]^{\frac{1}{4}}\left\{\left[\frac{\varepsilon(w_-(z,t))}{\mu(w_-(z,t))}\right]^{\frac{1}{4}}\vec{E}_0(w_-(z,t))+\right. \\ & \left[\frac{\varepsilon(w_+(z,t))}{\mu(w_+(z,t))}\right]^{\frac{1}{4}}\vec{E}_0(w_+(z,t))+ \\ & \left.\int_{w_-(z,t)}^{w_+(z,t)}dw\;\mu(w)^{\frac{1}{4}}\varepsilon(w)^{\frac{3}{4}}\dot{\vec{E}}_0(w)\right\}. \end{aligned} \qquad (27)$$

According to (22), $w_\pm(z,0)=z$. This shows that setting $t=0$, the integral on the right-hand side of (27) disappears and the remaining terms add up to give $\vec{E}_0(z)$. Similarly we can use (16) and (22) to establish $\dot{w}_\pm(z,0)=\pm v(z)=\pm[\varepsilon(z)\mu(z)]^{-\frac{1}{2}}$ and use the latter to show that setting $t=0$ in the time-derivative of the right-hand side of (27) yields $\dot{\vec{E}}_0(z)$.

In vacuum where $v=(\varepsilon\mu)^{-1/2}=c$, WKB approximation is exact, $u(z)=z/c$, $w_\pm(z,t)=z\pm ct$, and (27) gives

$$\begin{aligned}\vec{E}(z,t) \;=\; & \frac{1}{2}\left\{\vec{E}_0(z-ct)+\vec{E}_0(z+ct)+\right. \\ & \left.\frac{1}{c}\int_{z-ct}^{z+ct}dw\;\dot{\vec{E}}_0(w)\right\}, \end{aligned} \qquad (28)$$

which is precisely D'Alembert's solution of the 1+1 dimensional wave equation [10].

In general, (27) is a valid solution of Maxwell's equations, if WKB approximation is reliable. This is the case whenever $|Q(z)|$ is negligibly smaller than the right-hand side of (12), [18]. In view of (15) and (16) this condition takes the form

$$\frac{v^2}{2}\left|\frac{vv''-\frac{1}{2}\,v'^2}{v^2}+\frac{\mu\mu''-\frac{3}{2}\,\mu'^2}{\mu^2}\right|\ll\omega^2, \qquad (29)$$

where we have suppressed the z dependence of v and μ. Due to the asymptotic behavior of μ and ε, v tends to constant values as $z\to\pm\infty$. This in turn implies that the square root of the left-hand side of (29) has a least upper bound that we denote by $\omega_{\min}$. In this case, (29) means $|\omega|\gg\omega_{\min}$. Recalling the role of ω in our derivation of (27), we can view this condition as a restriction on the choice of the initial conditions. More specifically, (27) is a good approximation provided that for all $\omega\in[-\omega_{\min},\omega_{\min}]$, $\langle\psi_\omega|\varepsilon^{\frac{1}{2}}|\vec{E}_0(z)|\rangle\approx 0$ and $\langle\psi_\omega|\varepsilon^{\frac{1}{2}}|\dot{\vec{E}}_0(z)|\rangle\approx 0$. For a planar laser pulse with initial envelope functions $\mathcal{E}_0$ and $\mathcal{B}_0$ picked far away from the region where ε and μ vary significantly, these conditions hold for $c^{-1}\omega_{\min}\ll|k_0|$. The same is true for an initial plane wave with sufficiently large wave number $|k_0|$.

Next, we wish to elaborate on the relationship between our approach and the standard application of the WKB approximation in solving Maxwell equations, particularly for the effectively one-dimensional model we have been considering. The first step in the direct application of the WKB approximation to Maxwell equations is to assume a harmonic time-dependence for the solution. This yields an ordinary differential equation that can be treated using the WKB approximation, [9]. If we apply $\varepsilon^{-1/2}$ to eigenfunctions (17) of the Hermitian operator h we obtain (WKB-approximate) eigenfunctions of the pseudo-Hermitian operator Ω^2, [6]. It is very easy to check that these are solutions of the Maxwell equations (2) with a harmonic time-dependence. Up to a trivial normalization constant, they coincide with the conventional WKB solutions:

$$\vec{E}_\omega(z,t)=\left[\frac{\mu(z)}{\varepsilon(z)}\right]^{\frac{1}{4}}e^{i\omega t}e^{i\omega u(z)}.$$

Furthermore, if we make the additional assumption that $\vec{E}_\omega$ form a complete set in the sense that for any solution $\vec{E}(z,t)$ of Maxwell's equations, there are $c(\omega)\in\mathbb{C}$ such that

$$\vec{E}(z,t)=\int_{-\infty}^{\infty}d\omega\;c(\omega)\vec{E}_\omega(z,t), \qquad (30)$$

then we can use the initial electric and magnetic field to determine $c(\omega)$ and use the result to evaluate the integral in (30). The end result of this lengthy calculation is identical to (27).

The observations that Ω^2 is pseudo-Hermitian and that it can be mapped by a similarity transformation to a Hermitian operator h provide a simple justification for the above-mentioned assumptions regarding harmonic time-dependence and completeness of the conventional WKB solutions. The consequences of the pseudo-Hermiticity of Ω^2 is not limited to the application of the WKB approximation. For instance, it implies the existence of exact solutions of Maxwell's equations that have harmonic time-dependence and form a complete set.

As a concrete example of the application of (27), sup-

pose that ε has a Lorentzian shape and μ is a constant:

$$\varepsilon(z) = \varepsilon_0 \left[1 + a\left(1+\gamma^{-2}z^2\right)^{-1}\right], \qquad \mu = \mu_0, \quad (31)$$

where $\varepsilon_0, a, \gamma, \mu_0$ are positive constants and $\varepsilon_0\mu_0 = c^{-2}$. Then, by inspection, we can show that

$$\omega_{\min} \leq \frac{c\sqrt{3a(1+\nu a)}}{2\gamma},$$

where $\nu := 1 + 45/256 \approx 1.176$. For $a < 1$, $\omega_{\min} < c\gamma^{-1}$. This means that WKB approximation and consequently (27) are valid provided that the allowed ω values be much larger than $c\gamma^{-1}$.

In order to implement (27), we must compute $w_\pm$. This involves the evaluation and inversion of u. For the choice (31), we expand u and u^{-1} in power series in the inhomogeneity parameter a and perform a perturbative calculation of $w_\pm$. This yields

$$\begin{aligned} w_\pm(z,t) &= z \pm ct \mp a\left\{\frac{\gamma\theta_\pm(z,t)}{2}\right\} \pm a^2\Big\{\frac{\gamma}{16} \times \\ & [\lambda(z,t)\,\theta_\pm(z,t) + \nu_\pm(z,t)]\Big\} + \mathcal{O}(a^3), \quad (32) \end{aligned}$$

where

$$\begin{aligned} \theta_\pm(z,t) &:= \tan^{-1}\left(\frac{\gamma ct}{\gamma^2 + z(z\pm ct)}\right), \\ \lambda(z,t) &:= 1 + \frac{4\gamma^2}{\gamma^2 + (z\pm ct)^2}, \\ \nu_\pm(z,t) &:= \frac{\gamma ct[\gamma^2 - z(z\pm ct)]}{(\gamma^2+z^2)[\gamma^2+(z\pm ct)^2]}. \end{aligned}$$

In view of (31) and (32), we also have

$$\begin{aligned} \left[\frac{\varepsilon(w_\pm(z,t))}{\varepsilon(z)}\right]^{\frac{1}{4}} &= 1 \mp a\,\xi(z,t) \pm a^2\Big\{\xi(z,t) + \\ & \zeta(z,t) \mp \frac{3}{2}\,\xi(z,t)^2\Big\} + \mathcal{O}(a^3), \quad (33) \end{aligned}$$

where

$$\begin{aligned} \xi(z,t) &:= \frac{\gamma^2\,ct(2z\pm ct)}{4(\gamma^2+z^2)[\gamma^2+(z\pm ct)^2]}, \\ \zeta(z,t) &:= \frac{\gamma^3(z\pm ct)}{4[\gamma^2+(z\pm ct)^2]^2}. \end{aligned}$$

With the help of (32) and (33) we can use (27) to determine the dynamical behavior of the EM fields for initial configurations of the form (9) provided that the initial fields do not violate the condition of the reliability of the WKB approximation and that we can neglect the third and higher order contributions in powers of a. For example we can use this method to determine the effect of the inhomogeneity (31) on the planar propagation of a Gaussian laser pulse [19].

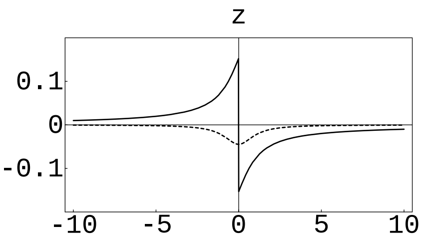

FIG. 1: Plots of Δr (full curve) and $\Delta\rho$ (dotted curve) as a function of z for $a = 0.2$ in units where $\gamma = 1$.

Another application of our results is in the solution of the scattering problem. It is not difficult to see that for $t \to \infty$ (i.e., $ct \gg z, \gamma$),

$$w_\pm(z,t) \to \left\{ \begin{array}{ll} z \pm ct + \Delta r(z) & \text{for} \quad z \neq 0 \\ \pm ct \pm \Delta r(0) & \text{for} \quad z = 0 \end{array} \right., \quad (34)$$

$$\left[\frac{\varepsilon(w_\pm(z,t))}{\varepsilon(z)}\right]^{\frac{1}{4}} \to \Delta\rho(z), \quad (35)$$

where

$$\begin{aligned} \Delta r(z) &:= -\frac{a\gamma}{2}\tan^{-1}\left(\frac{\gamma}{z}\right) + \\ & \frac{a^2\gamma}{16}\Big[\tan^{-1}\left(\frac{\gamma}{z}\right) - \frac{\gamma z}{\gamma^2+z^2}\Big] + \mathcal{O}(a^3), \\ \Delta\rho(z) &:= \left[\frac{\varepsilon_0}{\varepsilon(z)}\right]^{\frac{1}{4}} + \mathcal{O}(a^3) \\ &= 1 - \frac{a\gamma^2}{4(\gamma^2+z^2)} + \frac{5a^2\gamma^4}{32(\gamma^2+z^2)^2} + \mathcal{O}(a^3). \end{aligned}$$

According to (27), (34) and (35), the scattering of an initial plane wave, with $\vec{E}_0 = e^{-ik_0 z}\hat{\imath}$ and $\vec{B}_0 = \vec{0}$, by the inhomogeneity (31) results in a change in the amplitude and phase angle of the wave that are respectively given by $\Delta\rho(z)$ and $-k_0\Delta r(z)$. Specifically, as $t \to \infty$,

$$\vec{E}(z,t) \to \vec{E}_s(z,t) := \vec{E}(z,t)\Big|_{a=0}\,\Delta\rho(z)\,e^{-ik_0\Delta r(z)}, \quad (36)$$

where $\vec{E}(z,t)\big|_{a=0} := \frac{1}{2}\left(e^{-ik_0(z-ct)} + e^{-ik_0(z+ct)}\right)\hat{\imath}$. The predictions of (36) should be experimentally verifiable in typical interferometry experiments [20]. Figure 1 shows the plots of $\Delta\rho$ and Δr for $a = 0.2$. As seen from this figure Δr has a discontinuity at $z = 0$.

A quantity of direct physical relevance is the Fourier transform $\tilde{\vec{E}}_s(k,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{ikz}\vec{E}_s(z,t)$ of $\vec{E}_s(z,t)$. Up to linear terms in a, it is given by

$$\tilde{\vec{E}}_s(k,t) = \Big[\delta(k_0-k) + a\Delta\tilde{E}(k_0,k) + \mathcal{O}(a^2)\Big]\cos(k_0ct)\hat{\imath},$$

where $\Delta\tilde{E}(k_0,k) := \gamma\left[\frac{1-e^{-\gamma|k_0-k|}}{4(1-k/k_0)} - \frac{e^{-\gamma|k_0-k|}}{8}\right]$. Figure 2 shows the graph of $\Delta\tilde{E}(k_0,k)$ as a function of k. Again there is a discontinuity at $k = k_0$.

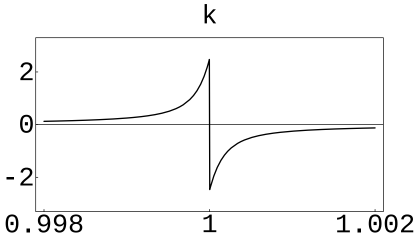

FIG. 2: Plot of $\Delta\tilde{E}(k_0,k)$ as a function of k for $\gamma = 10^{-3}m$ and $k_0 = 10^7 m^{-1}$. k is measured in units of $k_0 = 10^7 m^{-1}$.

To summarize, we have obtained a closed form expression (4) for the propagation of EM fields in an arbitrary non-dispersive stationary medium that yields the fields in terms of a pseudo-Hermitician operator Ω^2. This allows for formulating the problem in terms of an equivalent Hermitian operator h. Using the spectral resolution of h and the WKB approximation we have found an explicit formula, namely (27), for the propagating fields and demonstrated its application for the scattering of plane waves moving in an inhomogeneous non-dispersive medium. Although similar spectral techniques have previously been used in dealing with EM waves [11], we believe that our treatment provides a more straightforward and systematic solution for this problem. Indeed to our knowledge an expression for the propagating field as general and explicit as Eq. (27) has not appeared in the literature previously. Another important aspect of our method is the wide range of its applications, e.g., it can be used to study wave propagation in inhomogeneous fibers, observation of superfluid vortices, etc.

Our results may be generalized in various directions. For example, for the cases that $\overleftrightarrow{\varepsilon}$ fails to be Hermitian, one may appeal to the notion of weak pseudo-Hermiticity [12] and use the results of [13] to obtain an appropriate equivalent Hermitian operator h to Ω^2. One may also incorporate the dispersion effects by letting the $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}'$ that appear in the eigenvalue equation for h to depend (via a dispersion relation) on ω. This will lead to a modification of (27) that we plan to explore in future.

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[14] For other applications of pseudo-Hermitian operators see [4].
[15] Eliminating $\vec{E}$ in (2) yields $\ddot{\vec{B}} + \Gamma^2\vec{B} = 0$ where $\Gamma^2 := \mathfrak{D}\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'$. The solution of $\ddot{\vec{B}} + \Gamma^2\vec{B} = 0$ is $\vec{B} = \cos(t\Gamma)\vec{B}_0 + \Gamma^{-1}\sin(t\Gamma)\dot{\vec{B}}_0$. Applying $\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'$ to both sides of this equation and using (2) and the identity $\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'\Gamma^2 = \Omega^2\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'$, we obtain $\dot{\vec{E}} = -\Omega\sin(\Omega t)\vec{E}_0 + \cos(\Omega t)\dot{\vec{E}}_0$, which is consistent with (4).
[16] One might try to compute the terms in the derivative expansions appearing on the right-hand side of (4). Even for the simplest choices for the medium and the initial data, this yields a power series solution of Maxwell's equation in time whose summation proves to be extremely difficult if not impossible. Approximating the terms in this series and summing the corresponding approximate series seems to be also a formidable task.
[17] This is because $h = \mathfrak{a}^\dagger\mathfrak{a}$ where $\mathfrak{a} := \overleftrightarrow{\mu}'^{\frac{1}{2}}\mathfrak{D}\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}$.
[18] If $\varepsilon(z) = a\mu(z)[b + \int_0^z d\mathfrak{z}\,\mu(\mathfrak{z})^{-1}]^{-4}$ for some constants a and b, $Q(z) = 0$ and WKB approximation is exact.
[19] We have studied a pulse with $\mathcal{E}_0(z) = A\, e^{-\frac{(z+L)^2}{2\sigma^2}}$, $\mathcal{B}_0(z) = 0$, $A, L, \sigma \in \mathbb{R}^+$, $L \gg \gamma$, and $L \gg \sigma$. Then the WKB approximation is valid whenever $\gamma^{-1} \ll k_0$. We do not report the results here for lack of space.
[20] One can imagine using a Mach-Zender interferometer to detect the effect of the inhomogeneity in the interference pattern of two beams one travelling through the medium and the other through the vacuum.